# Relations and Functions

## Cartesian Product

* Let be sets, .
* An **ordered pair**  is a pair of elements with the property.
* NOTE: The open internal uses the same notation, but context makes it clear.
* The **Cartesian product** of and , denoted by , is the set of all ordered pairs with .

Exercise:

Let . What is ?



Exercise:

Let . What is ?



Exercise:

Let . What are and ?



Exercise:

Let . Is ?



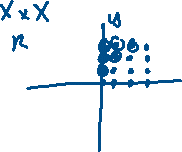
## Relations

* We say that is a **(binary) relation** from to if is a subset of .
* If , then is called a **relation of .**
* We say that is related to by if .
* This is denoted by .

Exercise:

Let

1. What is an easier way of expressing ?
2. List all the elements of .
3. Sketch and circle the elements of .



Exercise:

Let be given by .

1. Describe the relation .
2. True or false?



Exercise:



Let be given by

1. Give another description of .
2. Which are elements of ?
3. Prove that .



### Union and Intersection of Relations

* Relations are sets, so the set operations apply.

Exercise:

Let by given by .

Write expressions for and .



#### Definition (Domain and Range)

* Let be a relation from to .
* The **domain** of and the **range** of , denoted respectively by and , are defined:
* Note that and .

Exercise:

Let . Write and .



Exercise:

Find domain and range of .



Exercise:

Find domain and range of .



### The Inverse of a Relation

* If is on , then a relation on can be defined by interchanging the elements of the ordered pairs of .

#### Definition:

* Let be on . The inverse relation of is:
* Note that and .

Exercise:

Let . Find .



Exercise:

Define on by . Write 3 elements of and 3 elements of . Write a definition of .



Exercise:

The identity relation on is . What is ?



### Properties of Relations

* Let be a relation on . Then:

1. is **reflexive** on IFF .
2. is **symmetric** on IFF .
3. is **transitive** on IFF .

Exercise:

Which properties do the following relations satisfy?

1. On ,



1. On , the identity relation



1. On ,



1. On ,



1. On the set of all people,



1. On the set of all people,



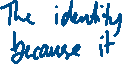
### Equivalence Relations

#### Definition

* Let be a relation on . Then is an equivalence relation of IFF is *reflexive*, *symmetric*, and *transitive* on .

Exercise:

Prove or disprove that the identity relation on is an equivalence relation.



Exercise:

On , prove that is an equivalence relation.



* To disprove an equivalence relation, you only need to show that one of the properties does not hold.

Exercise:

On , prove that is not an equivalence relation.



## Equivalence Classes

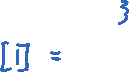
#### Definition

* Let be an equivalence relation on . For each , the **equivalence class** of , denoted , is the set:
* Equivalence classes have the following properties:

1. For any , we have either or .
2. All distinct equivalence classes form a **partition** of
   1. The *union* of all classes is , and the *intersection* of any 2 classes is empty.

Exercise:

Let . Find .



Exercise:

What do the equivalence classes of the identity relation on look like?



Exercise:

Let be defined by . Find .

